

On Carnot algebra with the growth vector $(2, 3, 5, 8)$

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April 1, 2013

Dedicated to Andrei Aleksandrovich Agrachev, with gratitude

Abstract

We compute two vector field models of the Carnot algebra with the growth vector $(2, 3, 5, 8)$, and an infinitesimal symmetry of the corresponding sub-Riemannian structure.

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1 Introduction

Carnot groups provide a nilpotent approximation to generic sub-Riemannian manifolds [8, 11, 3, 2, 5]. The free nilpotent sub-Riemannian structures are a natural first subject of study in sub-Riemannian geometry starting from the growth vector $(2, 3)$ — the left-invariant sub-Riemannian structure on the Heisenberg group [4, 16]. The next free rank 2 case — the growth vector $(2, 3, 5)$ — was studied in [12, 13, 14, 15].

In this work we start to study the next free rank 2 case — the growth vector $(2, 3, 5, 8)$. We compute two vector field models of the corresponding Carnot algebra, and an infinitesimal symmetry of the corresponding sub-Riemannian structure (see Sec. 3).

2 Free nilpotent and Carnot Lie algebras

2.1 Free nilpotent Lie algebras

Let \mathcal{L}_d be the real free Lie algebra with d generators [6]; \mathcal{L}_d is the Lie algebra of commutators of d variables. We have $\mathcal{L}_d = \bigoplus_{i=1}^{\infty} \mathcal{L}_d^i$, where \mathcal{L}_d^i is the space of commutator polynomials of degree i . Then

$$\mathcal{L}_d^{(r)} := \mathcal{L}_d / \bigoplus_{i=r+1}^{\infty} \mathcal{L}_d^i$$

is the free nilpotent Lie algebra of step r (or of length r).

Denote

$$l_d(i) := \dim \mathcal{L}_d^i, \quad l_d^{(r)} := \dim \mathcal{L}_d^{(r)} = \sum_{i=1}^r l_d(i).$$

The classical expression of $l_d(i)$ is

$$il_d(i) = d^i - \sum_{j|i, 1 \leq j < i} jl_d(j).$$

In this work we will be interested in free nilpotent Lie algebras with 2 generators. Dimensions of such Lie algebras for small step are given in Table 1.

2.2 Carnot algebras and groups

A Lie algebra L is called a Carnot algebra if it admits a decomposition

$$L = \bigoplus_{i=1}^r L_i$$

i	1	2	3	4	5	6	7	8	9	10
$l_2(i)$	2	1	2	3	6	9	18	30	56	99
$l_2^{(i)}$	2	3	5	8	14	23	41	71	127	226

Table 1: Dimensions of free nilpotent Lie algebras with 2 generators

as a vector space, such that

$$\begin{aligned} [L_i, L_j] &\subset L_{i+j}, \\ L_s &= 0 \text{ for } s > r, \\ L_{i+1} &= [L_1, L_i]. \end{aligned}$$

A free nilpotent Lie algebra $\mathcal{L}_d^{(r)}$ is a Carnot algebra with $L_i = \mathcal{L}_d^i$.

A Carnot group G is a connected, simply connected Lie group whose Lie algebra L is a Carnot algebra. If L is realized as the Lie algebra of left-invariant vector fields on G , then the degree 1 component L_1 can be thought of as a completely nonholonomic (bracket-generating) distribution on G . If moreover L_1 is endowed with a left-invariant inner product $\langle \cdot, \cdot \rangle$, then $(G, L_1, \langle \cdot, \cdot \rangle)$ becomes a nilpotent left-invariant sub-Riemannian manifold [5]. Such sub-Riemannian structures are nilpotent approximations of generic sub-Riemannian structures [8, 11, 3, 2].

The sequence of numbers

$$(\dim L_1, \dim L_1 + \dim L_2, \dots, \dim L_1 + \dots + \dim L_r = \dim L)$$

is called the growth vector of the distribution L_1 [16].

For free nilpotent Lie algebras, the growth vector is maximal compared with all Carnot algebras with the bidimension $(\dim L_1, \dim L)$.

In this work we consider the Carnot algebra with the growth vector $(2, 3, 5, 8)$.

3 Lie algebra with the growth vector $(2, 3, 5, 8)$

The Carnot algebra with the growth vector $(2, 3, 5, 8)$

$$\mathcal{L}_2^{(4)} = \text{span}(X_1, \dots, X_8)$$

is determined by the following multiplication table:

$$[X_1, X_2] = X_3, \tag{1}$$

$$[X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \tag{2}$$

$$[X_1, X_4] = X_6, \quad [X_1, X_5] = [X_2, X_4] = X_7, \quad [X_2, X_5] = X_8, \tag{3}$$

with all the rest brackets equal to zero.

3.1 Hall basis

Free nilpotent Lie algebras have a convenient basis introduced by M. Hall [9]. We describe it using the exposition of [7].

The Hall basis of the free Lie algebra \mathcal{L}_d with d generators X_1, \dots, X_d is the subset $\text{Hall} \subset \mathcal{L}_d$ that has a decomposition into homogeneous components

$$\text{Hall} = \cup_{i=1}^{\infty} \text{Hall}_i$$

defined as follows.

Each element H_j , $j = 1, 2, \dots$, of the Hall basis is a monomial in the generators X_i and is defined recursively as follows. The generators satisfy the inclusion

$$X_i \in \text{Hall}_1, \quad i = 1, \dots, d,$$

and we denote

$$H_i = X_i, \quad i = 1, \dots, d.$$

If we have defined basis elements

$$H_1, \dots, H_{N_{p-1}} \in \oplus_{j=1}^{p-1} \text{Hall}_j,$$

they are simply ordered so that $E < F$ if $E \in \text{Hall}_k$, $F \in \text{Hall}_l$, $k < l$:

$$H_1 < H_2 < \dots < H_{N_{p-1}}.$$

Also if $E \in \text{Hall}_s$, $F \in \text{Hall}_t$ and $p = s + t$, then

$$[E, F] \in \text{Hall}_p$$

if:

1. $E > F$, and
2. if $E = [G, K]$, then $K \in \text{Hall}_q$ and $t \geq q$.

By this definition, one easily computes recursively the first components Hall_i of the Hall basis for $d = 2$:

$$\begin{aligned} \text{Hall}_1 &= \{H_1, H_2\}, & H_1 &= X_1, & H_2 &= X_2, \\ \text{Hall}_2 &= \{H_3\}, & H_3 &= [X_2, X_1], \\ \text{Hall}_3 &= \{H_4, H_5\}, & H_4 &= [[X_2, X_1], X_1], & H_5 &= [[X_2, X_1], X_2], \\ \text{Hall}_4 &= \{H_6, H_7, H_8\}, \\ H_6 &= [[[X_2, X_1], X_1], X_1], & H_7 &= [[[X_2, X_1], X_1], X_2], & H_8 &= [[[X_2, X_1], X_2], X_2]. \end{aligned}$$

Consequently,

$$\mathcal{L}_2^{(4)} = \text{span}\{H_1, \dots, H_8\}.$$

In the sequel we use a more convenient basis

$$\mathcal{L}_2^{(4)} = \text{span}\{X_1, \dots, X_8\}$$

with the multiplication table (1)–(3).

3.2 Asymmetric vector field model for $\mathcal{L}_2^{(4)}$

Here we recall an algorithm for construction of a vector field model for the Lie algebra $\mathcal{L}_2^{(r)}$ due to Grayson and Grossman [7]. For a given $r \geq 1$, the algorithm evaluates two polynomial vector fields $H_1, H_2 \in \text{Vec}(\mathbb{R}^N)$, $N = \dim \mathcal{L}_2^{(r)}$, which generate the Lie algebra $\mathcal{L}_2^{(r)}$.

Consider the Hall basis elements

$$\text{span}\{H_1, \dots, H_N\} = \mathcal{L}_2^{(r)}.$$

Each element $H_i \in \text{Hall}_j$ is a Lie bracket of length j :

$$H_i = [\dots [[H_2, H_{k_j}], H_{k_{j-1}}], \dots, H_{k_1}], \\ k_j = 1, \quad k_{n+1} \leq k_n \text{ for } 1 \leq n \leq j-1.$$

This defines a partial ordering of the basis elements. We say that H_i is a direct descendant of H_2 and of each H_{k_l} and write $i \succ 2$, $i \succ k_l$, $l = 1, \dots, j$.

Define monomials $P_{2,k}$ in x_1, \dots, x_N inductively by

$$P_{2,k} = -x_j P_{2,i} / (\deg_j P_{2,i} + 1),$$

whenever $H_k = [H_i, H_j]$ is a basis Hall element, and where $\deg_j P$ is the highest power of x_j which divides P .

The following theorem gives the properties of the generators.

Theorem 3.1 (Th. 3.1 [7]). *Let $r \geq 1$ and let $N = \dim \mathcal{L}_2^{(r)}$. Then the vector fields*

$$H_1 = \frac{\partial}{\partial x_1}, \quad H_2 = \frac{\partial}{\partial x_2} + \sum_{i \succ 2} P_{2,i} \frac{\partial}{\partial x_i}$$

have the following properties:

1. *they are homogeneous of weight one with respect to the grading*

$$\mathbb{R}^N = \text{Hall}_1 \oplus \dots \oplus \text{Hall}_r;$$

$$2. \text{ Lie}(H_1, H_2) = \mathcal{L}_2^{(r)}.$$

The algorithm described before Theorem 3.1 produces the following vector field basis of $\mathcal{L}_2^{(4)}$:

$$\begin{aligned} H_1 &= \frac{\partial}{\partial x_1}, \\ H_2 &= \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} - \frac{x_1^2}{2} \frac{\partial}{\partial x_4} - x_1 x_2 \frac{\partial}{\partial x_5} + \frac{x_1^3}{6} \frac{\partial}{\partial x_6} + \frac{x_1^2 x_2}{2} \frac{\partial}{\partial x_7} + \frac{x_1 x_2^2}{2} \frac{\partial}{\partial x_8}, \\ H_3 &= \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5} - \frac{x_1^2}{2} \frac{\partial}{\partial x_6} - x_1 x_2 \frac{\partial}{\partial x_7} - \frac{x_2^2}{2} \frac{\partial}{\partial x_8}, \\ H_4 &= -\frac{\partial}{\partial x_4} + x_1 \frac{\partial}{\partial x_6} + x_2 \frac{\partial}{\partial x_7}, \\ H_5 &= -\frac{\partial}{\partial x_5} + x_1 \frac{\partial}{\partial x_7} + x_2 \frac{\partial}{\partial x_8}, \\ H_6 &= -\frac{\partial}{\partial x_6}, \\ H_7 &= -\frac{\partial}{\partial x_7}, \\ H_8 &= -\frac{\partial}{\partial x_8}, \end{aligned}$$

with the multiplication table

$$[H_2, H_1] = H_3, \tag{4}$$

$$[H_3, H_1] = H_4, [H_3, H_2] = H_5, \tag{5}$$

$$[H_4, H_1] = H_6, [H_4, H_2] = H_7, [H_5, H_2] = H_8. \tag{6}$$

3.3 Symmetric vector field model of $\mathcal{L}_2^{(4)}$

The vector field model of the Lie algebra $\mathcal{L}_2^{(4)}$ via the fields H_1, \dots, H_8 obtained in the previous subsection is asymmetric in the sense that there is no visible symmetry between the vector fields H_1 and H_2 . Moreover, no continuous symmetries of the sub-Riemannian structure generated by the orthonormal frame $\{H_1, H_2\}$ are visible, although the Lie brackets (4)–(6) suggest that this sub-Riemannian structure should be preserved by a one-parameter group of rotations in the plane $\text{span}\{H_1, H_2\}$.

One can find a symmetric vector field model of $\mathcal{L}_2^{(4)}$ free of such shortages as in the following statement.

Theorem 3.2. (1) *The vector fields*

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2} \frac{\partial}{\partial x_5} - \frac{x_1 x_2^2}{4} \frac{\partial}{\partial x_7} - \frac{x_2^3}{6} \frac{\partial}{\partial x_8}, \quad (7)$$

$$X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3} + \frac{x_1^2 + x_2^2}{2} \frac{\partial}{\partial x_4} + \frac{x_1^3}{6} \frac{\partial}{\partial x_6} + \frac{x_1^2 x_2}{4} \frac{\partial}{\partial x_7}, \quad (8)$$

$$X_3 = \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5} + \frac{x_1^2}{2} \frac{\partial}{\partial x_6} + x_1 x_2 \frac{\partial}{\partial x_7} + \frac{x_2^2}{2} \frac{\partial}{\partial x_8}, \quad (9)$$

$$X_4 = \frac{\partial}{\partial x_4} + x_1 \frac{\partial}{\partial x_6} + x_2 \frac{\partial}{\partial x_7}, \quad (10)$$

$$X_5 = \frac{\partial}{\partial x_5} + x_1 \frac{\partial}{\partial x_7} + x_2 \frac{\partial}{\partial x_8}, \quad (11)$$

$$X_6 = \frac{\partial}{\partial x_6}, \quad (12)$$

$$X_7 = \frac{\partial}{\partial x_7}, \quad (13)$$

$$X_8 = \frac{\partial}{\partial x_8} \quad (14)$$

satisfy the multiplication table (1)–(3). Thus the fields $X_1, \dots, X_8 \in \text{Vec}(\mathbb{R}^8)$ model the Lie algebra $\mathcal{L}_2^{(4)}$.

(2) *The vector field*

$$X_0 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_5 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_5} + P \frac{\partial}{\partial x_6} + Q \frac{\partial}{\partial x_7} + R \frac{\partial}{\partial x_8}, \quad (15)$$

$$P = -\frac{x_1^4}{24} + \frac{x_1^2 x_2^2}{8} + x_7, \quad (16)$$

$$Q = \frac{x_1 x_2^3}{12} + \frac{x_1^3 x_2}{12} - 2x_6 + 2x_8, \quad (17)$$

$$R = \frac{x_1^2 x_2^2}{8} - \frac{x_2^4}{24} - x_7 \quad (18)$$

satisfies the following relations:

$$[X_0, X_1] = X_2, \quad [X_0, X_2] = -X_1, \quad [X_0, X_3] = 0, \quad (19)$$

$$[X_0, X_4] = X_5, \quad [X_0, X_5] = -X_4, \quad (20)$$

$$[X_0, X_6] = 2X_7, \quad [X_0, X_7] = X_8 - X_6, \quad [X_0, X_8] = -2X_7. \quad (21)$$

Thus the field X_0 is an infinitesimal symmetry of the sub-Riemannian structure generated by the orthonormal frame $\{X_1, X_2\}$.

Proof. In fact, the both statements of the proposition are verified by the direct computation, but we prefer to describe a method of construction of the vector fields X_1, \dots, X_8 , and X_0 .

(1) In the previous work [12] we constructed a similar symmetric vector field model for the Lie algebra $\mathcal{L}_2^{(3)}$, which has growth vector $(2, 3, 5)$:

$$\mathcal{L}_2^{(3)} = \text{span}\{X_1, \dots, X_5\} \subset \text{Vec}(\mathbb{R}^5), \quad (22)$$

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2} \frac{\partial}{\partial x_5}, \quad (23)$$

$$X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3} + \frac{x_1^2 + x_2^2}{2} \frac{\partial}{\partial x_4}, \quad (24)$$

$$X_3 = \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5}, \quad (25)$$

$$X_4 = \frac{\partial}{\partial x_4}, \quad (26)$$

$$X_5 = \frac{\partial}{\partial x_5}, \quad (27)$$

with the Lie brackets (1), (2). Now we aim to “continue” these relationships to vector fields $X_1, \dots, X_8 \in \text{Vec}(\mathbb{R}^8)$ that span the Lie algebra $\mathcal{L}_2^{(4)}$. So we seek for vector fields of the form

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2} \frac{\partial}{\partial x_5} + \sum_{i=6}^8 a_1^i \frac{\partial}{\partial x_i}, \quad (28)$$

$$X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2} \frac{\partial}{\partial x_4} + \sum_{i=6}^8 a_2^i \frac{\partial}{\partial x_i}, \quad (29)$$

$$X_3 = \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5} + \sum_{i=6}^8 a_3^i \frac{\partial}{\partial x_i}, \quad (30)$$

$$X_4 = \frac{\partial}{\partial x_4} + \sum_{i=6}^8 a_4^i \frac{\partial}{\partial x_i}, \quad (31)$$

$$X_5 = \frac{\partial}{\partial x_5} + \sum_{i=6}^8 a_5^i \frac{\partial}{\partial x_i}, \quad (32)$$

$$X_j = \sum_{i=i}^j a_i^j \frac{\partial}{\partial x_j}, \quad j = 6, 7, 8, \quad (33)$$

such that $\text{span}\{X_1, \dots, X_8\} = \mathcal{L}_2^{(4)}$.

Compute the required Lie brackets:

$$\begin{aligned}
[X_1, X_2] &= \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5} + \left(\frac{\partial a_2^6}{\partial x_1} - \frac{\partial a_1^6}{\partial x_2} \right) \frac{\partial}{\partial x_6} \\
&\quad + \left(\frac{\partial a_2^7}{\partial x_1} - \frac{\partial a_1^7}{\partial x_2} \right) \frac{\partial}{\partial x_7} + \left(\frac{\partial a_2^8}{\partial x_1} - \frac{\partial a_1^8}{\partial x_2} \right) \frac{\partial}{\partial x_8}, \\
[X_1, X_3] &= \frac{\partial}{\partial x_4} + \frac{\partial a_3^6}{\partial x_1} \frac{\partial}{\partial x_6} + \frac{\partial a_3^7}{\partial x_1} \frac{\partial}{\partial x_7} + \frac{\partial a_3^8}{\partial x_1} \frac{\partial}{\partial x_8}, \\
[X_2, X_3] &= \frac{\partial}{\partial x_5} + \frac{\partial a_3^6}{\partial x_2} \frac{\partial}{\partial x_6} + \frac{\partial a_3^7}{\partial x_2} \frac{\partial}{\partial x_7} + \frac{\partial a_3^8}{\partial x_2} \frac{\partial}{\partial x_8}, \\
[X_1, X_4] &= \frac{\partial a_4^6}{\partial x_1} \frac{\partial}{\partial x_6} + \frac{\partial a_4^7}{\partial x_1} \frac{\partial}{\partial x_7} + \frac{\partial a_4^8}{\partial x_1} \frac{\partial}{\partial x_8}, \\
[X_1, X_5] &= \frac{\partial a_5^6}{\partial x_1} \frac{\partial}{\partial x_6} + \frac{\partial a_5^7}{\partial x_1} \frac{\partial}{\partial x_7} + \frac{\partial a_5^8}{\partial x_1} \frac{\partial}{\partial x_8}, \\
[X_2, X_4] &= \frac{\partial a_4^6}{\partial x_2} \frac{\partial}{\partial x_6} + \frac{\partial a_4^7}{\partial x_2} \frac{\partial}{\partial x_7} + \frac{\partial a_4^8}{\partial x_2} \frac{\partial}{\partial x_8}, \\
[X_2, X_5] &= \frac{\partial a_5^6}{\partial x_2} \frac{\partial}{\partial x_6} + \frac{\partial a_5^7}{\partial x_2} \frac{\partial}{\partial x_7} + \frac{\partial a_5^8}{\partial x_2} \frac{\partial}{\partial x_8}.
\end{aligned}$$

The vector fields X_1, \dots, X_8 should be independent, thus the determinant constructed of these vectors as columns should satisfy the inequality

$$D = \det(X_1, \dots, X_8) = \begin{vmatrix} a_6^6 & a_7^6 & a_8^6 \\ a_6^7 & a_7^7 & a_8^7 \\ a_6^8 & a_7^8 & a_8^8 \end{vmatrix} \neq 0.$$

We will choose a_i^j such that $D = 1$. It follows from the multiplication table for X_1, \dots, X_8 that

$$D = \begin{vmatrix} \frac{d^2 a_3^6}{dx_1^2} & \frac{d^2 a_3^6}{dx_1 dx_2} & \frac{d^2 a_3^6}{dx_2^2} \\ \frac{d^2 a_3^7}{dx_1^2} & \frac{d^2 a_3^7}{dx_1 dx_2} & \frac{d^2 a_3^7}{dx_2^2} \\ \frac{d^2 a_3^8}{dx_1^2} & \frac{d^2 a_3^8}{dx_1 dx_2} & \frac{d^2 a_3^8}{dx_2^2} \end{vmatrix}.$$

In order to get $D = 1$, define the entries of this matrix as following symmetric way:

$$a_3^6 = \frac{x_1^2}{2}, \quad a_3^7 = x_1 x_2, \quad a_3^8 = \frac{x_2^2}{2}.$$

Then we obtain from the multiplication table for X_1, \dots, X_8 that

$$\begin{aligned}\frac{\partial a_2^6}{\partial x_1} - \frac{\partial a_1^6}{\partial x_2} &= a_3^6 = \frac{x_1^2}{2}, \\ \frac{\partial a_2^7}{\partial x_1} - \frac{\partial a_1^7}{\partial x_2} &= a_3^7 = x_1 x_2, \\ \frac{\partial a_2^8}{\partial x_1} - \frac{\partial a_1^8}{\partial x_2} &= a_3^8 = \frac{x_2^2}{2}.\end{aligned}$$

We solve these equations in the following symmetric way:

$$\begin{aligned}a_1^6 &= 0, & a_2^6 &= \frac{x_1^3}{6}, \\ a_1^7 &= -\frac{x_1 x_2^2}{4}, & a_2^7 &= \frac{x_1^2 x_2}{4}, \\ a_1^8 &= -\frac{x_2^3}{6}, & a_2^8 &= 0.\end{aligned}$$

Then we substitute these coefficients to (28), (29) and check item (1) of this theorem by direct computation.

Now we prove item (2). We proceed exactly as for item (1): we start from an infinitesimal symmetry [12]

$$X_0 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_5 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_5} \in \text{Vec}(\mathbb{R}^5) \quad (34)$$

of the sub-Riemannian structure on \mathbb{R}^5 determined by the orthonormal frame (23), (24) and “continue” symmetry (34) to the sub-Riemannian structure on \mathbb{R}^8 determined by the orthonormal frame (7), (8).

So we seek for a vector field $X_0 \in \text{Vec}(\mathbb{R}^8)$ of the form (15) for the functions $P, Q, R \in C^\infty(\mathbb{R}^8)$ to be determined so that the multiplication table (19)–(21) hold.

The first two equalities in (19) yield

$$X_1 P = -\frac{x_1^3}{6}, \quad X_2 P = \frac{x_1^2 x_2}{2}.$$

Further,

$$X_3 P = [X_1, X_2] P = X_1 X_2 P - X_2 X_1 P = X_1 \frac{x_1^2 x_2}{2} + X_2 \frac{x_1^3}{6} = x_1 x_2.$$

Similarly it follows that

$$X_4 P = x_2, \quad X_5 P = x_1, \quad X_6 P = 0, \quad X_7 P = 1, \quad X_8 P = 0.$$

Since $X_6P = X_8P = 0$, then $P = P(x_1, x_2, x_3, x_4, x_5, x_7)$. Moreover, since $X_7P = 1$, then $P = x_7 + a(x_1, x_2, x_3, x_4, x_5)$. The equality $X_5P = x_1$ implies that $\frac{\partial a}{\partial x_5} = 0$, i.e., $a = a(x_1, x_2, x_3, x_4)$. Similarly, since $X_4P = x_2$, then $a = a(x_1, x_2, x_3)$. It follows from the equality $X_3P = x_1x_2$ that $\frac{\partial a}{\partial x_3} = x_1x_2$, i.e., $a = x_1x_2x_3 + b(x_1, x_2)$. Moreover, the equality $X_2P = \frac{x_1^2x_2}{2}$ implies that $\frac{\partial b}{\partial x_2} = -x_1x_3 - \frac{x_1^2x_2}{4}$, i.e., $b = -x_1x_2x_3 - \frac{x_1^2x_2^2}{8} + c(x_1)$. Finally, the equality $X_1P = -\frac{x_1^3}{2}$ implies that $\frac{dc}{dx_1} = -\frac{x_1^3}{6} + \frac{x_1x_2^2}{2}$ i.e., $c = -\frac{x_1^4}{24} + \frac{x_1^2x_2^2}{4}$. Thus equality (16) follows. Similarly we get equalities (17), (18).

Then multiplication table (19)–(21) for the vector field (15)–(18) is verified by a direct computation. \square

4 Conclusion and future work

We plan to perform a further study of the nilpotent sub-Riemannian structure with the growth vector $(2, 3, 5, 8)$ using its model obtained in Th. 3.2:

- describe the multiplication rule on the corresponding Carnot group \mathbb{R}^8 ,
- characterize Casimir functions and orbits of the co-adjoint action,
- describe abnormal extremal trajectories and prove strict abnormality of some of them,
- study symmetries, integrals and integrability of the Hamiltonian system for normal extremals,
- describe normal extremal trajectories.

These results will be published elsewhere.

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